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The modified mild-slope equation

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A modified version of the mild-slope equation is derived and its predictions of wave scattering by two-dimensional topography compared with those of other equations and with experimental data. In particular, the modified mild-slope equation is shown to be capable of describing known scattering properties of singly and doubly periodic ripple beds, for which the mild-slope equation fails. The new equation compares favourably with other models of scattering which improve on the mild-slope equation, in that it is widely applicable and computationally cheap.

1. Introduction

The scattering of linear water waves by bed topography is governed by Laplace's equation together with appropriate boundary and radiation conditions. Analytic solutions are rare when there is any departure from the constant depth case and usually exist only for simple geometries involving horizontal and/or vertical boundaries. Consequently, a number of approximations to the boundary value problem have been proposed. In one class of approximations the vertical coordinate is removed by integration over the depth, thus reducing the dimension of the problem by one. Berkhoff's mild-slope equation (1973, 1976) is the result of such a procedure, and seeks to approximate the propagating wave mode.

It has been observed by a number of authors that the mild-slope equation can fail to produce adequate approximations for certain types of topography, such as ripple beds. These consist of a finite patch of small-amplitude sinusoidal ripples set in an otherwise horizontal bed. It is perhaps more accurate to refer to the bed perturbations as bars rather than ripples, but we follow what has become the standard terminology. To overcome the deficiency in the mild-slope equation, Kirby (1986) presented a model in which the bed profile consists of a slowly varying (mild-slope) component on which is superimposed a rapidly varying component of small amplitude. Applying the vertical integration process led Kirby to what is now called the extended mild-slope equation, which he verified for ripple beds by comparing numerical results with wave-tank data of Davies & Heathershaw (1984).

Another approximation which has proved successful for ripple bed problems is the 'successive-application matrix model' of O'Hare & Davies (1993), in which the topography is replaced by a succession of short horizontal steps. The scattering properties of the whole topography are then approximated by aggregating the scattering properties of the individual steps. More recently, Massel (1993) has proposed a new approximation which includes evanescent modes and is therefore capable of dealing with relatively steep bed profiles. Guazzelli, Rey & Belzons (1992) also include evanescent modes in conjunction with a step-wise approximation of the bed, and produce theoretical results in good agreement with experimental data for doubly periodic sinusoidal beds.

In the present paper we return to the relatively simple type of approximation used by Berkhoff (1973, 1976) and Kirby (1986) and present a new form of the mild-slope equation. This equation, which contains as special cases the usual mild-slope equation and Kirby's extended mild-slope equation, was first derived by Chamberlain (1991). We refer to it as the modified mild-slope equation.

Two related derivations of the new equation are given. The first exploits a variational principle and is similar in approach to the recent work of Miles (1991). The second derivation is a direct application of the classical Galerkin method, which evidently formalizes the vertical integration method used by Berkhoff (1973, 1976) and others. Our approach clearly distinguishes the two approximations which together lead to the mild-slope equation. We invoke only one of these approximations, the replacement of the velocity potential by a one-term trial function based on the propagating wave mode over a flat bed. We do not, however, take the further step of discarding terms which are second-order on the basis of the mild-slope assumption $|\nabla h| \ll kh$, where h is the undisturbed fluid depth and k is the corresponding wavenumber.

One result of retaining all of the terms produced by the Galerkin approximation is that the modified mild-slope equation accurately predicts wave scattering by ripple beds. A simplified form of the new equation, derived in § 3, may be compared with Kirby's approximation (1986), and is computationally more efficient for calculating scattering by ripple beds.

2. The modified mild-slope equation

We suppose that incompressible, homogeneous fluid is in irrotational motion over a bed of varying quiescent depth $h(x,y)$, x and y denoting horizontal Cartesian coordinates. The vertical coordinate, z, is measured positively upwards with the undisturbed free surface at $z=0$.

An harmonic time dependence can be removed from the velocity potential Φ which describes the fluid motion, by setting

$$\Phi(x,y,z,t) = \operatorname{Re}\left(\phi(x,y,z)e^{-i\sigma t}\right),$$

where σ is an assigned angular frequency. Then the function ϕ satisfies the usual equations of linearized wave theory, namely

$$\nabla^2\phi = 0 \quad (-h < z < 0), \tag{2.1}$$

$$\phi_z - \nu\phi = 0 \quad (z=0), \tag{2.2}$$

$$\phi_z + \nabla_h h \cdot \nabla_h \phi = 0 \quad (z=-h), \tag{2.3}$$

where $\nu = \sigma^2/g$, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and $\nabla_h = (\partial/\partial x, \partial/\partial y)$. The free surface elevation is given by $\zeta(x,y,t) = \operatorname{Re}\left(\eta(x,y)e^{-i\sigma t}\right)$ where

$$\eta(x,y) = \frac{i\sigma}{g}(\phi)_{z=0}. \tag{2.4}$$

The specification of ϕ is completed by the addition of conditions on lateral boundaries or a radiation condition if the fluid extends to infinity. These further conditions do not concern us for the present as our immediate objective is to reduce the dimension of the boundary value problem for ϕ by approximating its dependence on the z-coordinate.

We present two equivalent ways of achieving the desired approximation.

First we consider the variational principle $\delta L = 0$ where L is the functional defined by

$$L(\psi) = \iint_D \left(\frac{1}{2} \nu (\psi^2)_{z=0} - \frac{1}{2} \int_{-h}^{0} (\nabla\psi)^2 \mathrm{d}z \right) \mathrm{d}x\mathrm{d}y.$$

Here D denotes a domain in the plane $z = 0$, with boundary C. As indicated above, we need not specify D more closely for present purposes and we may consider variations which vanish on the lateral boundary $C \times [-h, 0]$. It then follows that L is stationary at $\psi = \phi$ if and only if ϕ satisfies (2.1), (2.2) and (2.3).

The variational principle $\delta L = 0$ can therefore be used to generate approximations to solutions of (2.1), (2.2) and (2.3). In particular, we can seek the one-term approximation $\psi \approx \phi$ of the form

$$\psi(x, y, z) = \phi_0(x, y) w(x, y, z), \tag{2.5}$$

where w is a given function and ϕ_0 is determined by imposing $\delta L = 0$ for all variations in ϕ_0 which vanish on $C \times [-h, 0]$. After some straightforward manipulation it is found that ϕ_0 must satisfy

$$\nabla_h \cdot \int_{-h}^{0} w^2 \mathrm{d}z \nabla_h \phi_0 + \left(\int_{-h}^{0} w w_{zz} \mathrm{d}z + R(w) \right) \phi_0 = 0, \tag{2.6}$$

where

$$R(w) = \int_{-h}^{0} w \nabla_h^2 w \mathrm{d}z - [w(w_z - \nu w)]_{z=0} + [w(w_z + \nabla_h h \cdot \nabla_h w)]_{z=-h}.$$

The association between variational principles and Galerkin's method suggests an alternative derivation of (2.6). Suppose we set aside (2.5) for the moment and seek a weak solution $\psi \approx \phi$ of (2.1) in the sense that the residual $\nabla^2\psi$ is required to be orthogonal to a given function w. Thus

$$\begin{aligned}&\iint_D \int_{-h}^{0} w \nabla^2 \psi \mathrm{d}z\mathrm{d}x\mathrm{d}y \\ &= \iint_D \left(\int_{-h}^{0} (w \nabla_h^2 \psi + \psi w_{zz}) \mathrm{d}z + [w\psi_z - \psi w_z]_{z=-h}^{z=0} \right) \mathrm{d}x\mathrm{d}y = 0\end{aligned}$$

which gives

$$\begin{aligned}&\iint_D \Bigg(\int_{-h}^{0} (w \nabla_h^2 \psi + \psi w_{zz}) \mathrm{d}z \\ &\quad - [\psi(w_z - \nu w)]_{z=0} + [\psi w_z + w \nabla_h h \cdot \nabla_h \psi]_{z=-h} \Bigg) \mathrm{d}x\mathrm{d}y = 0\end{aligned} \tag{2.7}$$

when the boundary conditions (2.2) and (2.3) are imposed on ψ. Equation (2.7) is a weak form of the boundary value problem (2.1), (2.2) and (2.3) and can be used to generate any desired approximation $\psi \approx \phi$. In particular, we can return to (2.5) and seek the particular Galerkin approximation $\psi = \phi_0 w$. Use of the identity $w\nabla_h^2(\phi_0 w) = \nabla_h \cdot w^2 \nabla_h \phi_0 + \phi_0 w \nabla_h^2 w$ followed by some manipulation leads again to the equation (2.6) for ϕ_0, whatever the domain D.

The development so far holds for any w, but we now make a particular choice of this function. Following the corresponding step in the derivation of the mild-slope equation we take

$$w = \mathrm{i}\sigma w_0/g, \qquad w_0(x,y,z) = \operatorname{sech}(kh)\cosh(k(z+h)) \tag{2.8}$$

where the local wavenumber $k = k(x,y)$ is the real, positive root of the local dispersion relation

$$v = k\tanh(kh) \tag{2.9}$$

corresponding to the depth $h(x,y)$. Over a flat bed, $\phi = w_0(z)\phi_0(x,y)$ is the only solution of (2.1), (2.2) and (2.3) which corresponds to propagating surface waves. The choice (2.8) with (2.9) ensures that the approximate solution $\phi \approx \phi_0 w$ satisfies the free surface condition and is such that $\phi_0 \approx \eta$, the complex form of the free surface elevation, by virtue of (2.4).

It is convenient at this point to note that (2.9) defines the function $k = k(h)$, at each fixed value of the angular frequency σ. This allows us to rewrite w_0 in the form

$$w_0(h,z) = \operatorname{sech}(kh)\cosh(k(z+h)). \tag{2.10}$$

We also introduce the function

$$u_0(h) = \int_{-h}^{0} w_0^2 \mathrm{d}z = \frac{1}{2k}\tanh(kh)\left(1 + \frac{2kh}{\sinh(2kh)}\right), \tag{2.11}$$

so that gu_0 is the product of the local phase velocity σ/k and the local group velocity $\mathrm{d}\sigma/\mathrm{d}k$, where the depth is h.

The result of substituting (2.8) into (2.6) may then be written in the form

$$\nabla_h \cdot u_0 \nabla_h \phi_0 + (k^2 u_0 + r)\phi_0 = 0 \tag{2.12}$$

where $r = R(w_0)$ is given by

$$r(h) = \int_{-h}^{0} w_0 \nabla_h{}^2 w_0 \mathrm{d}z + \nabla_h h \cdot [w_0 \nabla_h w_0]_{z=-h}.$$

The equation (2.12) coincides with the familiar mild-slope equation if the term $r(h)$ is omitted. The deletion of r is usually justified by noting that $r = O(|\nabla_h h|^2, \nabla_h{}^2 h)$ (see (2.15) below), which is assumed to be a negligibly small term on the basis of the mild-slope approximation.

However, we do not need to make the mild-slope approximation in the sense of neglecting r. As our derivation of (2.12) makes clear, the two approximations $\phi \approx \phi_0 w$ and $r \approx 0$ are essentially independent and we may suppose that the retention of r widens the scope of (2.12), which we refer to as the modified mild-slope equation. This description acknowledges that the use of (2.12) is still confined to slowly varying topography, because of the approximation $\phi \approx \phi_0 w$. More accurate approximations, of the form $\phi \approx \sum_{n=0}^{N} \phi_n w_n$ for $N > 0$, are required to remove the mild-slope restriction.

Miles (1991) showed how an existing variational principle for (nonlinear) free surface flows, due to Luke (1967), can be modified to apply to linearized free surface problems. Miles then used the modified variational principle (which differs from the present $\delta L = 0$ in that it is specific to a real-valued potential) to derive the mild-slope equation, evidently by discarding a term corresponding to r in the process.

Other derivations of the mild-slope equation, notably those by Berkhoff (1973, 1976), use an approximation equivalent to $\phi \approx \phi_0 w$ together with averaging over the fluid depth to remove the dependence on z. This vertical averaging procedure may be identified with the Galerkin formulation outlined above, although it is not usually presented in this direct way.

In order to use (2.12) in numerical calculations, it is convenient to evaluate r explicitly. The evaluation is expedited by deducing from (2.9) that

$$k'(h) = -2k^2(2kh + \sinh(2kh))^{-1} \tag{2.13}$$

and from (2.10) that

$$\begin{aligned} w_1(h,z) &= \frac{\partial w_0(h,z)}{\partial h} \\ &= k'(h)\operatorname{sech}(kh)\left(z\sinh(k(z+h)) - k^{-1}\sinh(kh)\sinh(kz)\right). \end{aligned} \tag{2.14}$$

It follows that $\nabla_h w_0 = w_1 \nabla_h h$ and hence that

$$w_0 \nabla_h{}^2 w_0 = w_1 w_0 \nabla_h{}^2 h + \nabla_h h \cdot \nabla_h(w_1 w_0) - w_1^2(\nabla_h h)^2,$$

leading to

$$r(h) = u_1(h)\nabla_h{}^2 h + u_2(h)(\nabla_h h)^2, \tag{2.15}$$

where

$$u_1(h) = \int_{-h}^{0} w_1 w_0 \mathrm{d}z, \qquad u_2(h) = u_1'(h) - \int_{-h}^{0} w_1^2 \mathrm{d}z. \tag{2.16}$$

These integrals are easily evaluated to give

$$u_1(h) = \frac{\operatorname{sech}^2(kh)}{4(K + \sinh(K))}\left\{\sinh(K) - K\cosh(K)\right\},$$

$$\begin{aligned} u_2(h) = \frac{k\operatorname{sech}^2(kh)}{12(K+\sinh(K))^3}\big\{&K^4 + 4K^3\sinh(K) - 9\sinh(K)\sinh(2K) \\ &+ 3K(K + 2\sinh(K))(\cosh^2(K) - 2\cosh(K) + 3)\big\}, \end{aligned}$$

where the abbreviation $K = 2kh$ has been used.

3. Ripple beds

The scattering of water waves by a finite patch of small-amplitude sinusoidal ripples set in an otherwise horizontal bed has recently received a good deal of attention. As such ripple beds can fall outside the scope of the mild-slope equation, Kirby (1986) derived an alternative equation which allowed for a rapidly varying, small-amplitude bedform to be superimposed on a slowly varying component of topography.

When applied to ripple beds, Kirby's extended mild-slope equation has the advantage that the wavenumber does not vary with the ripples and only one solution $k(h)$ of (2.9) is required, namely that corresponding to the mean still-water depth of the ripples. In contrast the wavenumber $k(h)$ in the modified mild-slope equation follows the topography. We therefore derive a simpler, and computationally cheaper, version of (2.12) which may be applied to ripple beds.

It is convenient, for the purpose of comparison, to adopt a notation very similar to that of Kirby (1986), first by writing $\tilde{h}$ in place of h and then setting

$$\tilde{h} = h - \delta. \tag{3.1}$$

Here $h = h(x,y)$ represents a slowly varying component of the depth, in the sense of the mild-slope assumption, and $\delta(x,y)$ represents a small-amplitude, rapidly-varying oscillation about the depth h. We therefore discard terms $O(\nabla_h^2 h, |\nabla_h h|^2)$ in this section, in accordance with the usual mild-slope approximation, as well as those $O(\delta^2)$.

Using (2.10), (2.11) and (2.14) we have

$$\begin{aligned} u_0(\tilde{h}) &= \int_{-h+\delta}^{0} w_0^2(h-\delta, z)\mathrm{d}z \\ &= u_0(h) - \delta \frac{\partial}{\partial h} \int_{-h}^{0} w_0^2(h,z)\mathrm{d}z + O(\delta^2) \\ &= u_0(h) - \delta \left(2 \int_{-h}^{0} w_0(h,z) w_1(h,z)\mathrm{d}z + \mathrm{sech}^2(kh) \right) + O(\delta^2) \\ &= u_0(h) - \delta \left(2u_1(h) + \mathrm{sech}^2(kh) \right) + O(\delta^2) \end{aligned}$$

where the notation of (2.16) has been employed. We also have $k(\tilde{h}) = k(h) - \delta k'(h) + O(\delta^2)$, where k' is given by (2.13), and, from (2.15), $r(\tilde{h}) = u_1(h)\nabla_h^2\delta + O(\nabla_h^2 h, |\nabla_h h|^2, \delta^2)$. Replacing h by $h - \delta$ in (2.12), and using the above expansions together with the identity $2k'(h)u_0(h) + k(h)\mathrm{sech}^2(kh) = 0$, we find that

$$\nabla_h \cdot (u_0 - \delta(2u_1 + \mathrm{sech}^2(kh)))\nabla_h \phi_0 + (k^2(u_0 - 2\delta u_1) - u_1 \nabla_h^2 \delta)\phi_0 = 0, \tag{3.2}$$

neglecting the terms indicated above, where u_0, u_1 and k are evaluated at h. In particular, these quantities are constant for ripple beds, where h is a constant and δ represents the only depth variation.

The version of the usual mild-slope equation which corresponds to (3.2) is obtained by deleting the term arising from $r(\tilde{h})$ and is therefore

$$\nabla_h \cdot (u_0 - \delta(2u_1 + \mathrm{sech}^2(kh)))\nabla_h \phi_0 + k^2(u_0 - 2\delta u_1)\phi_0 = 0. \tag{3.3}$$

Kirby's extended mild-slope equation, expressed in the present notation, is

$$\nabla_h \cdot u_0 \nabla_h \phi_0 - \mathrm{sech}^2(kh)\nabla_h \cdot \delta \nabla_h \phi_0 + k^2 u_0 \phi_0 = 0.$$

This can be written as

$$\nabla_h \cdot (u_0 - \delta \mathrm{sech}^2(kh))\nabla_h \phi_0 + k^2 u_0 \phi_0 = 0 \tag{3.4}$$

if terms $O(\delta \nabla_h h)$ are neglected, an approximation consistent with Kirby's derivation.

To see how (3.4) fits into the present framework, we first note that (3.2) can be derived directly from (2.6), rather than via (2.10). This is achieved if we first replace $-h$ with $-h + \delta$ in the integration limits and in the bed evaluation term of (2.6) and set $w = w_0(h - \delta, z)$. Expanding in powers of δ and neglecting terms $O(\nabla_h^2 h, |\nabla_h h|^2, \delta^2)$ reduces the resulting equation to (3.2). If this procedure is repeated with $w = w_0(h, z)$ used in place of $w = w_0(h - \delta, z)$, we merely omit the term u_1, which arises from the $O(\delta)$ correction $w_1 = \partial w_0/\partial h$ in w_0, and we arrive therefore at (3.4) instead of (3.2). This hybrid process of replacing h by $h - \delta$ only in selected terms of (2.6) is not as inconsistent as it may appear to be. It corresponds to the usual derivation of the mild-slope equation but with the bed condition modified at the outset to account for variations δ about h, and this is indeed the basis of Kirby's derivation.

4. Numerical experiments and discussion

For the purpose of comparison with existing results we consider the scattering of plane harmonic waves normally incident on a given bed profile $h = h(x)$. We assume that

$$h(x) = \begin{cases} h_0 & \forall x \leqslant 0, \\ h_1 & \forall x \geqslant L, \end{cases}$$

where h_0 and h_1 are given constants and L is also assigned.

In these circumstances, all of the model equations under consideration can be written in the form

$$(u\phi_0')' + v\phi_0 = 0, \tag{4.1}$$

where $\mathrm{Re}(\phi_0(x)\mathrm{e}^{-\mathrm{i}\sigma t})$ is an approximation to the surface elevation; the prime here denotes differentiation with respect to x. The functions $u(x)$ and $v(x)$ are such that (4.1) reduces to $\phi_0'' + k^2\phi_0 = 0$ where h is a constant and we therefore take

$$\phi_0(x) = \begin{cases} \mathrm{e}^{\mathrm{i}k_0 x} + R_\ell \mathrm{e}^{-\mathrm{i}k_0 x} & (x < 0), \\ T_\ell \mathrm{e}^{\mathrm{i}k_1 x} & (x > L), \end{cases}$$

for a wave of unit amplitude incident from the left and

$$\phi_0(x) = \begin{cases} T_r \mathrm{e}^{-\mathrm{i}k_0 x} & (x < 0), \\ \mathrm{e}^{-\mathrm{i}k_1 x} + R_r \mathrm{e}^{\mathrm{i}k_1 x} & (x > L), \end{cases}$$

for a wave of unit amplitude incident from the right, where $k_0 = k(h_0)$ and $k_1 = k(h_1)$. The complex amplitudes R_ℓ, R_r of the reflected waves and T_ℓ, T_r of the transmitted waves are the quantities of prime interest here.

They are obtained by solving (4.1) in the interval $(0, L)$, for a given $h(x)$ and a choice of u and v corresponding to the wave model being examined, and enforcing continuity of ϕ_0 and ϕ_0' at $x = 0$ and $x = L$. Details of the solution procedure employed may be found in Chamberlain & Porter (1995) and we merely describe the main features here.

We express the solution of (4.1) for $0 < x < L$ in terms of two linearly independent solutions of the equation, each generated by an initial value problem. This device detatches the essential structure of the problem from the numerical calculations, which are thereby reduced to the simplest possible form. The required values R_ℓ, R_r, T_ℓ and T_r are then given in terms of boundary values of the two computed linearly independent solutions.

A separate aspect of calculating the scattered wave amplitudes involves the notion of topography decomposition, introduced by Chamberlain (1995) and extended by Chamberlain & Porter (1995). Thus, the complete topography on $(0, L)$ may be decomposed into arbitrary sections and the overall scattered wave amplitudes determined from those of the individual sections. The process of combining the scattered wave amplitudes for two contiguous sections to form the net scattered wave amplitudes in general requires a knowledge of left- and right-wave scattering, even if one of these is not needed in the overall solution.

Decomposition is at its most powerful when dealing with periodic topography such as ripple beds, for then the overall scattering properties of N ripples can be assembled from a knowledge of the scattering characteristics of one ripple. Using $R_\ell^{(N)}$ to denote the reflected wave amplitude for a left-incident wave on a sequence of N ripples, and so on, Chamberlain & Porter (1995) show that

$$R_\ell^{(N)} = R_\ell^{(1)} \sin(N\theta) \left\{ \sin(N\theta) - T_\ell^{(1)} \sin(N-1)\theta \right\}^{-1},$$

$$\left| R_\ell^{(N)} \right|^2 = \left| R_\ell^{(1)} \right|^2 \sin^2(N\theta) \left\{ \left| R_\ell^{(1)} \right|^2 \sin^2(N\theta) + \left| T_\ell^{(1)} \right|^2 \sin^2\theta \right\}^{-1}$$

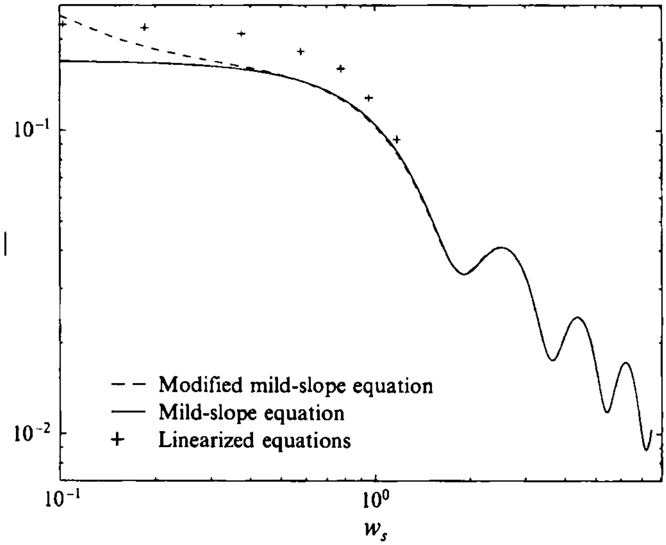

FIGURE 1. Comparison of computed reflection coefficients for Booij's test problem.

with corresponding formulae for the other wave amplitudes. Here the complex number θ is defined by

$$\cos(\theta) = \cos(\arg(T_\ell^{(1)})) \left| T_\ell^{(1)} \right|^{-1}.$$

In addition to the obvious computational saving which they produce, these formulae allow the mechanism which produces resonance, referred to below, to be identified. The details of this analysis may be found in Chamberlain & Porter (1995).

4.1. *Booij's problem*

In order to examine the range of validity of the mild-slope equation, Booij (1983) considered the scattering of plane waves incident normally on a depth profile in which the still water depth is reduced by a factor of one-third across a plane sloping section (i.e. $h_1 = h_0/3$ and $h(x) = h_0(1 - 2x/3L)$ for $x \in [0, L]$). Using the finite element method, Booij found approximations for the reflected amplitude $|R|$ corresponding to the mild-slope equation and also computed values of $|R|$ corresponding to the linearized problem consisting of (2.1), (2.2), (2.3) and appropriate radiation conditions. A comparison of these two sets of data led Booij to conclude that the mild-slope equation is valid for slopes of 1 in 3 and less (i.e., in our notation, for values of $\sigma^2 L/g > 1.2$). Here we compare the modified mild-slope equation (2.12) with the mild-slope equation ((2.12) with the term r deleted) for Booij's test problem. Figure 1 shows this comparison in terms of $|R|$ plotted against $w_s = \sigma^2 L/g$. The approximation obtained using the mild-slope equation is shown as a solid line while the modified approximation is given as a broken line; Booij's results for the linearized problem described above are shown as crosses on the graph. In the range where the two curves differ significantly the approximation obtained using the modified mild-slope equation is the closer to the solution of the linearized equations. This indicates that retaining the term r in (2.12) does indeed extend the range of validity of the model equation.

4.2. *Ripple beds*

As mentioned in the Introduction, it was the failure of the mild-slope equation to predict scattering by ripple beds accurately which provoked the derivation of other approximation methods. In particular, certain resonances, referred to below, were inadequately described by the mild-slope equation. We therefore examine the performance of the modified mild-slope equation in relation to ripple beds.

The first problem to be examined is defined on setting

$$h(x) = h_0, \qquad \delta(x) = d\sin(\ell x) \qquad (0 \leqslant x \leqslant L),$$

(in the notation of § 3) where h_0 is a constant and $L = 2n\pi/\ell$. The bedform therefore consists of a sequence of n sinusoidal ripples about the mean depth $z = -h_0$. Note that when using the modified mild-slope equation or the mild-slope equation (models not specifically designed with ripple beds in mind) the depth function we use there is the total depth $h - \delta$ rather than the two components h and δ considered separately.

The results presented are in the form of graphs of $|R|$ plotted against $2k/\ell$, this being twice the ratio of the wavenumber of the incident wave k and the ripple wavenumber ℓ. We present five graphs for each of the problems considered. Each graph corresponds to a different approximation and the labelling is as follows:

(i) the mild-slope equation (that is, equation (2.12) with the term r omitted);

(ii) the modified mild-slope equation, given by equation (2.12);

(iii) the approximate version of the mild-slope equation derived in § 3 and given by equation (3.3);

(iv) the approximate version of the modified mild-slope equation derived in § 3 and given by equation (3.2);

(v) Kirby's extended mild-slope equation given in § 3 by equation (3.4).

We recall that the model equations (iii) and (iv) are considered for two reasons. They are directly comparable with Kirby's equation (v) as all three are based on perturbations about the mean depth of the ripple; and they are computationally cheaper, as only the wavenumber corresponding to the mean depth is required.

However it should be noted that, by using a decomposition method (see Chamberlain 1995 or Chamberlain & Porter 1995), it is only necessary to approximate $|R|$ for the $n = 1$ case since it is then possible to infer the corresponding approximations to $|R|$ for larger n. The computational cost is therefore of no real significance.

Figure 2 shows results for the case $d/h_0 = 0.32$ and $n = 4$, for which wave-tank data are available by virtue of experiments carried out by Davies & Heathershaw (1984). The results of these experiments are shown on all five graphs in figure 2 as dots. In this case it is clear that all five models give good agreement at the first resonant peak near $2k/\ell = 1$. Only the models based on evaluating the local wavenumber at the actual depth $h - \delta$ (models (i) and (ii)) detect significant second-order resonance near $2k/\ell = 2$. Here we use the terms second- and higher-order in the sense described by Guazzelli *et al.* (1992).

Figure 3 gives graphs for the same problem, but with $d/h_0 = 0.16$ and $n = 10$. This is another of the cases considered by Davies & Heathershaw in their experiments. It is clear for these parameter values that the mild-slope equation and its approximated version, while correctly positioning the first-order resonance, completely fail to predict its magnitude. The other three models do describe that peak well but only the modified mild-slope equation detects any significant second-order resonance.

The perturbation procedure which replaced (2.12) with (3.2) and the mild-slope equation with (3.3) is well supported in these two figures. At almost all points on

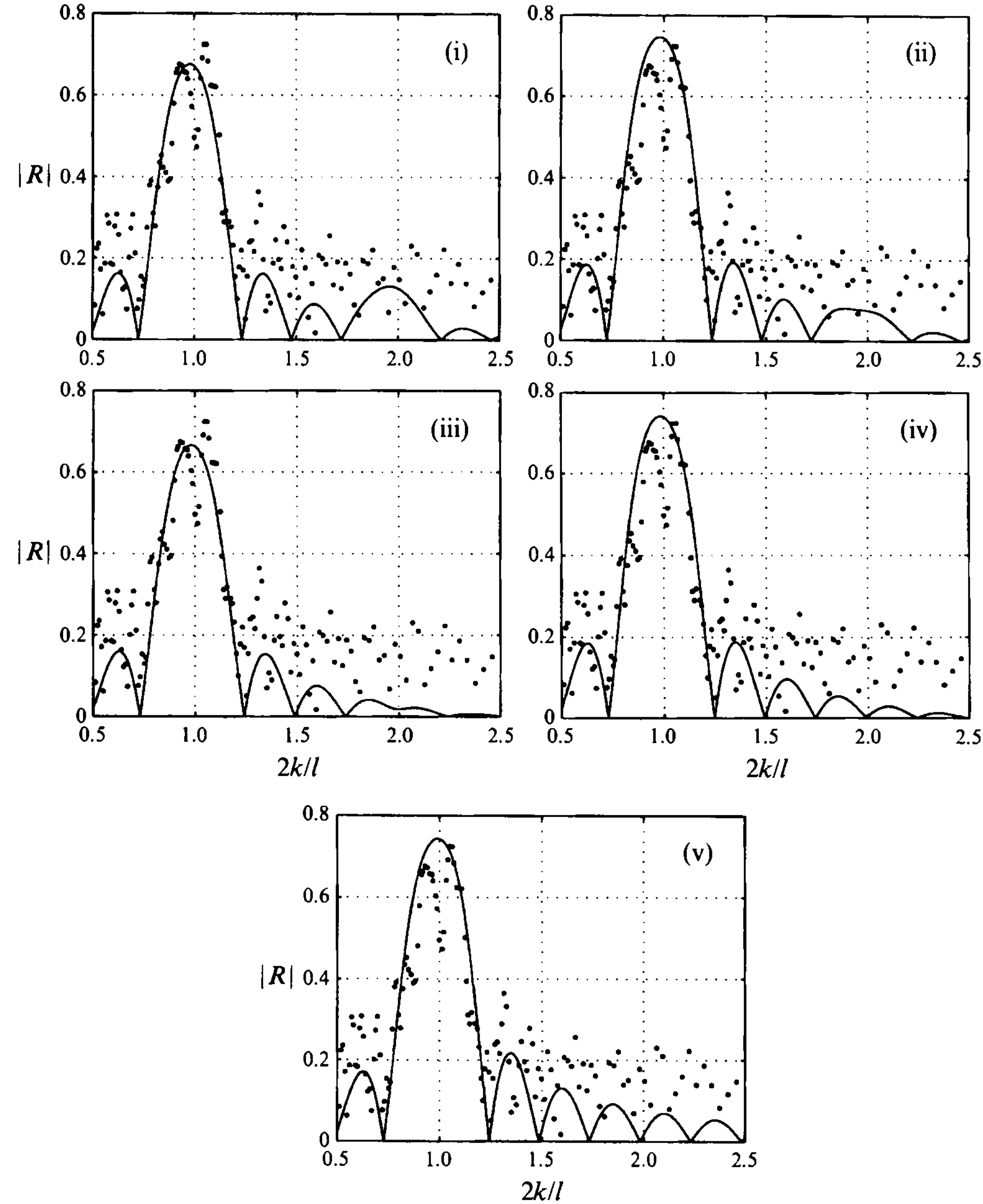

FIGURE 2. Comparison of computed reflection coefficients for singly periodic sinusoidal ripple with $n = 4$ and $d/h_0 = 0.32$. Corresponding experimental data produced by Davies & Heathershaw (1984) are shown as •.

both figures it is not possible to distinguish between the two curves with the eye. The principal exception to this observation occurs for the modified equation near $2k/\ell = 2$, where second-order resonance occurs. None of the models based on using a perturbation method detect significant second-order resonance at $k = \ell$. To $O(\delta^2)$ (the order to which equations (3.2), (3.3) and (3.4) are accurate) there appears only to be first-order resonance. A detailed analysis of how the resonant peaks depend on n may be found in Chamberlain & Porter (1995).

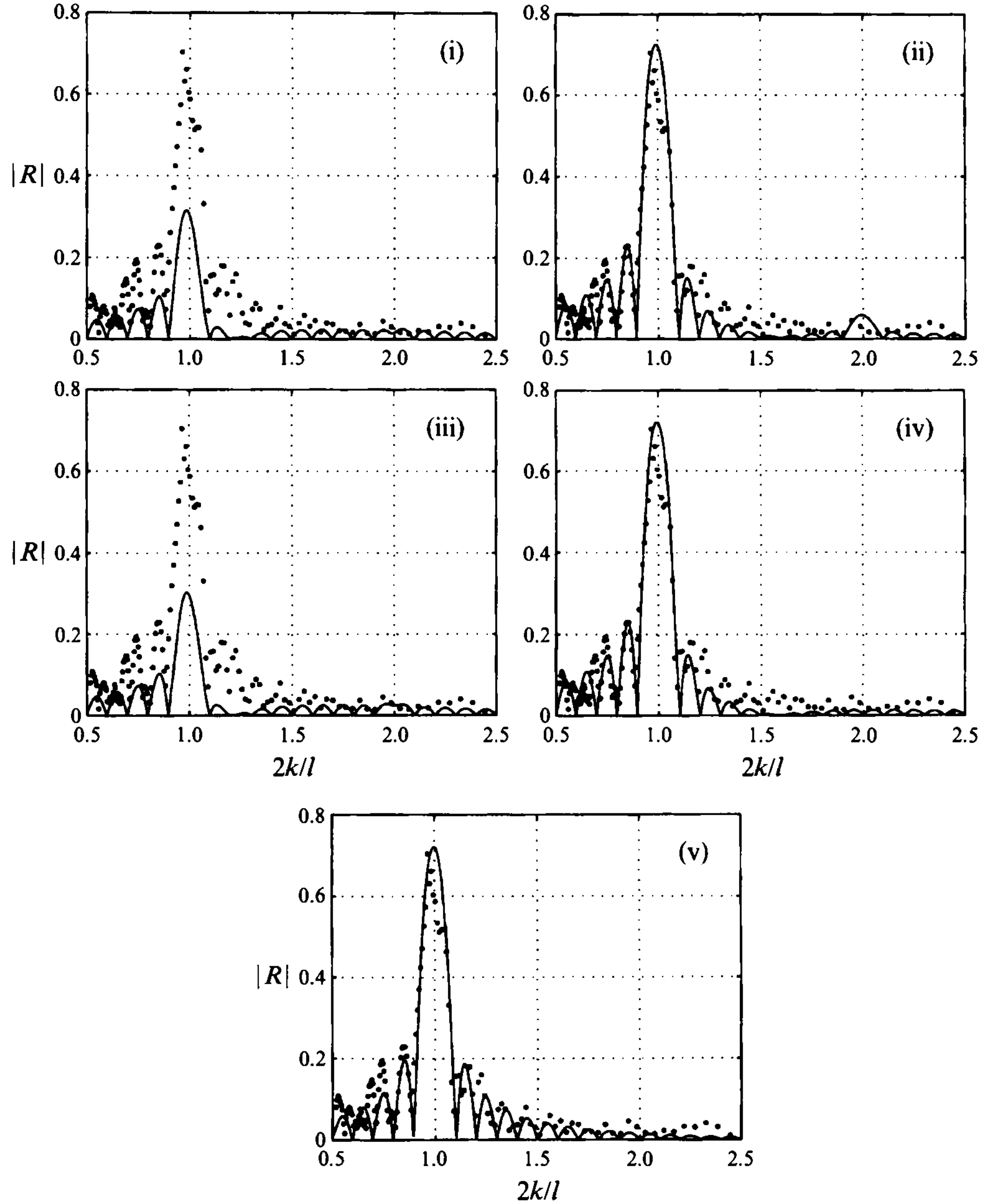

FIGURE 3. Comparison of computed reflection coefficients for singly periodic sinusoidal ripple with $n = 10$ and $d/h_0 = 0.16$. Corresponding experimental data produced by Davies & Heathershaw (1984) are shown as •.

This feature suggests that problems in which second- (or higher-) order resonance are significant may shed more light on the current examination of the modified equation. A suitable class of problems from the point of view of exhibiting higher-order resonance is known to be that involving ripples with two Fourier components, in which

$$\delta(x) = d(\sin(\ell x) + \sin(m\ell x)) \qquad (0 \leqslant x \leqslant L),$$

with $L = 2n\pi/\ell$ as before.

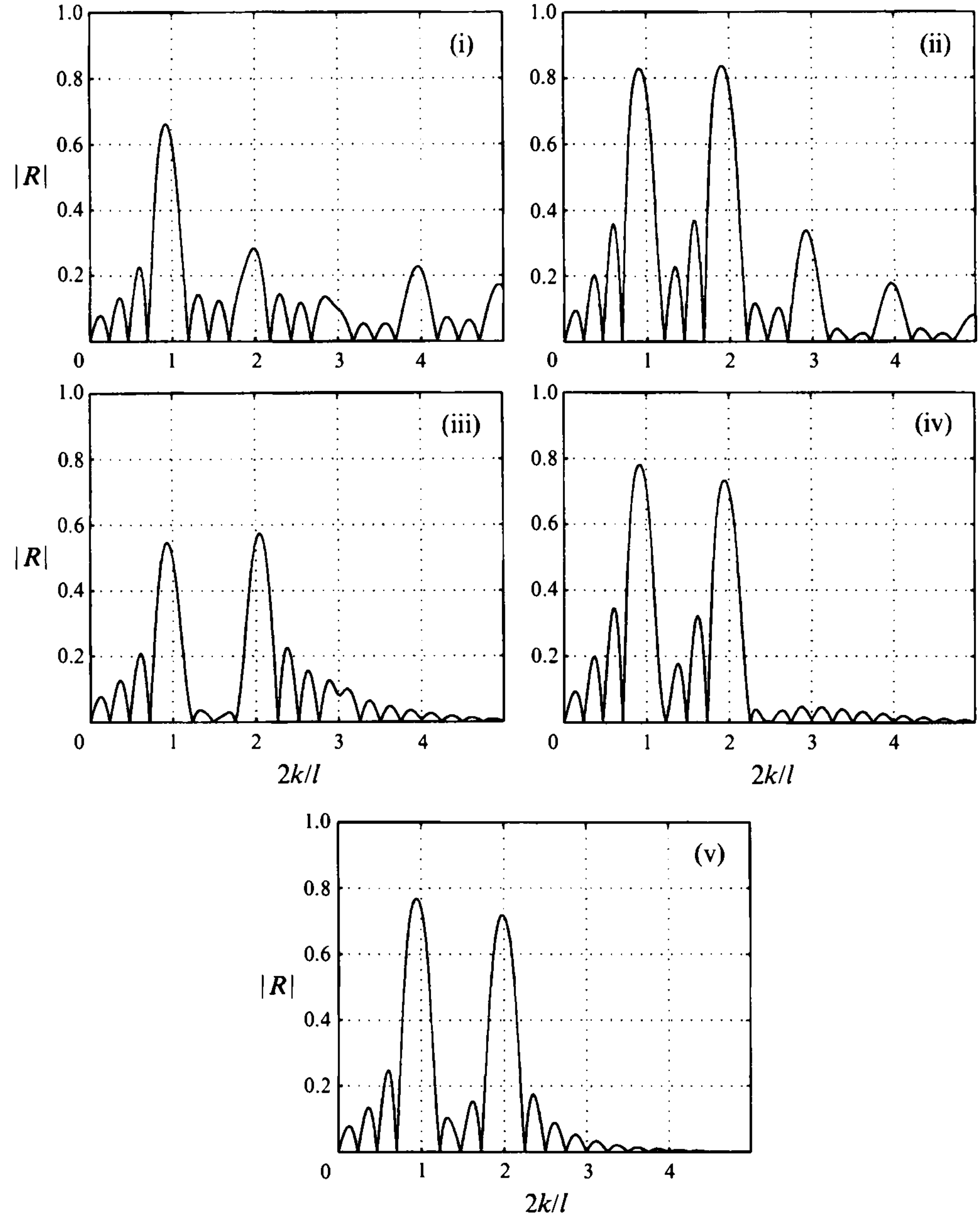

FIGURE 4. Comparison of computed reflection coefficients for doubly periodic sinusoidal ripple with $n = 4$, $m = 2$ and $d/h_0 = 0.33$.

This situation has been examined by O'Hare & Davies (1993). There exists the possibility of first-order resonances at $2k/\ell = 1$ and at $2k/\ell = m$, and at second order there are possible resonances due to the individual terms in δ at $2k/\ell = 2$ and at $2k/\ell = 2m$ as well as subharmonic (or difference) resonances at $2k/\ell = m - 1$ and harmonic (or sum) resonances at $2k/\ell = m + 1$. These last two resonances are due to interactions between the two terms in δ.

Figure 4 shows values of $|R|$ in the case $d/h_0 = 0.33$, $m = 2$ and $n = 4$. This problem was considered by Guazzelli *et al.* (1992) who carried out numerical and wave-

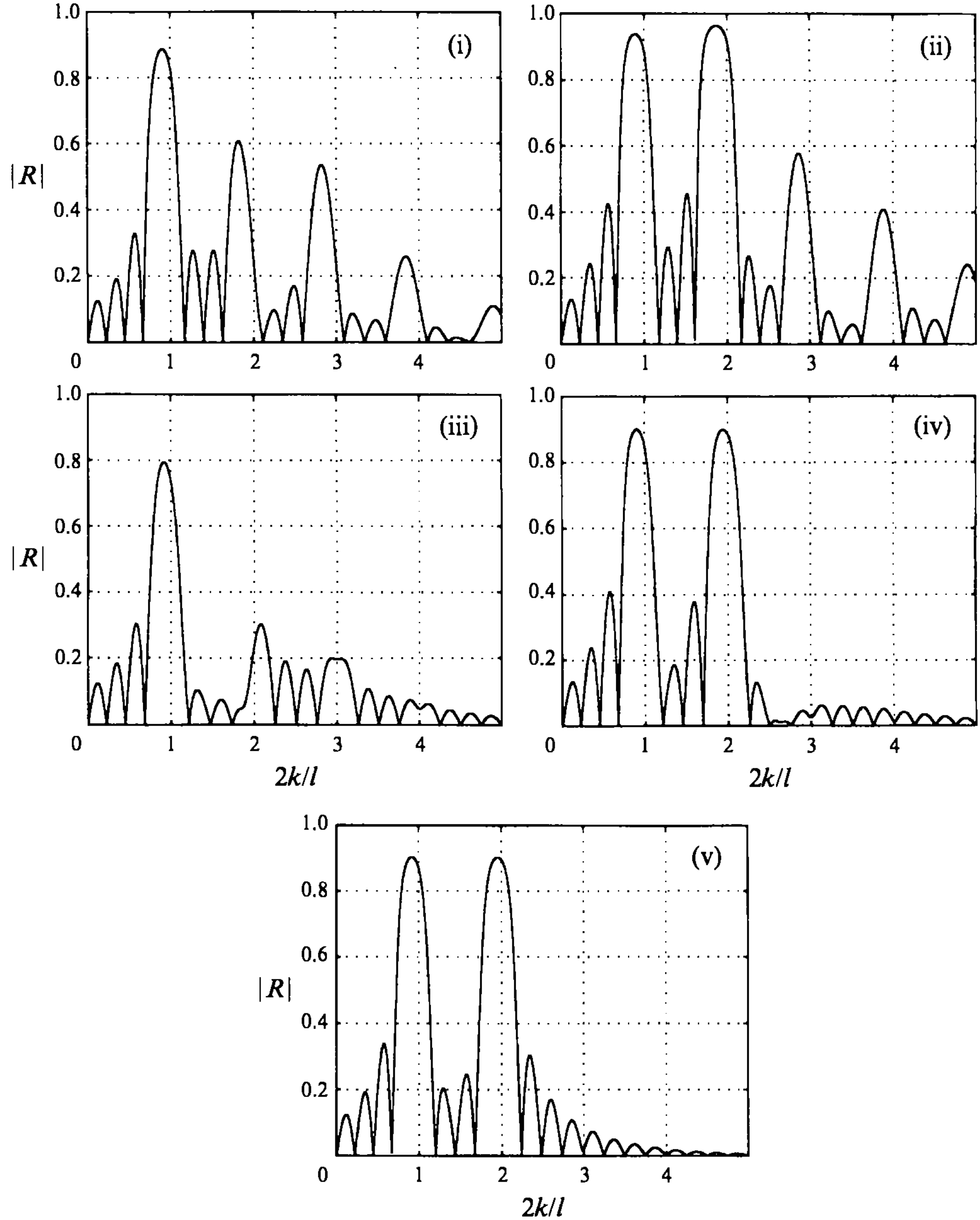

FIGURE 5. Comparison of computed reflection coefficients for doubly periodic sinusoidal ripple with $n = 4$, $m = 2$ and $d/h_0 = 0.4$.

tank experiments and by O'Hare & Davies (1993) who concentrated on numerical experiments, principally to demonstrate their successive-application matrix model. Once again we show five graphs in the arrangement described above. A comparison between figure 4 and the corresponding figure in O'Hare & Davies' paper (figure 5(b)) shows that the modified mild-slope equation produces approximations very similar to the matrix model and thus constitutes an improvement, not only to the mild-slope equation, but also to Kirby's extended mild-slope equation. In particular, the heights of the resonant peaks and the shift in their positions towards smaller values of $2k/\ell$

correspond to those given by the calculations of O'Hare & Davies (1993). It is also clear from such a comparison that the two mild-slope models ((i) and (iii)) are relatively unsuccessful for this problem. It is of interest to note that the models (iv) and (v) produce very similar graphs for this problem, both failing to detect second- or higher-order resonant peaks. The underestimated peak at $2k/\ell = 2$ may well be due to failure to model the second-order contribution from the first-order peak at $2k/\ell = 1$. Similarly the underestimated peak at $2k/\ell = 1$ may be due to the second-order subharmonic contribution being poorly approximated.

Finally, in figure 5, we consider the same doubly sinusoidal ripple bed problem with $d/h_0 = 0.4$, $m = 2$ and $n = 4$. This case was also considered by the authors referred to in the preceding paragraph and the results are qualitatively the same as for the parameter values considered there. O'Hare & Davies state that their model for this case predicts peaks near $2k/\ell = 3$ and $2k/\ell = 4$ which are too large. Comparing their figure for this problem (figure 5(c)) with the present figure 5 shows that use of the modified mild-slope equation goes some way to rectify this deficiency, by reducing the predicted peaks.

The important fact to note here is that the modified mild-slope equation produces results which are evidently as good as those given by much more complicated models of the scattering process. The modified mild-slope equation is merely the mild-slope equation with an additional term inserted and its solutions can therefore be approximated very efficiently without the need to assemble and solve large linear systems of equations. In this respect it contrasts sharply with the methods of O'Hare & Davies (1993) and Guazzelli *et al.* (1992). One particular advantage of predicting scattering fairly accurately by means of a single equation is that analytic investigations are possible. Further work is in progress on the modified mild-slope equation in this direction.

5. Conclusions

Much attention has recently been given to improving the mild-slope equation, which has proved to be deficient for certain bed profiles. A modified mild-slope equation has been derived here in two different, but related, ways. By retaining a term which has previously been discarded, this new equation successfully predicts known scattering phenomena which are undetected by Berkhoff's long-standing mild-slope equation.

One particular solution method which has been proposed for wave scattering by topography and which has proved successful is that in which the bed profile is replaced with a piecewise constant function, matching of solutions being carried out at the depth discontinuities (see Rey 1992 or O'Hare & Davies 1993, for example). This type of method is computationally expensive (owing to the number of steps required for reliable results) and can require the inclusion of the most significant evanescent modes to smooth the approximation near each step. In contrast, approximating the solution of the modified mild-slope equation can be very efficient computationally (see Chamberlain & Porter 1995) and permits analysis of the scattering process.

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